Elliptic Curve Cryptography

Cryptography

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elliptic.tex, r1949

Contents

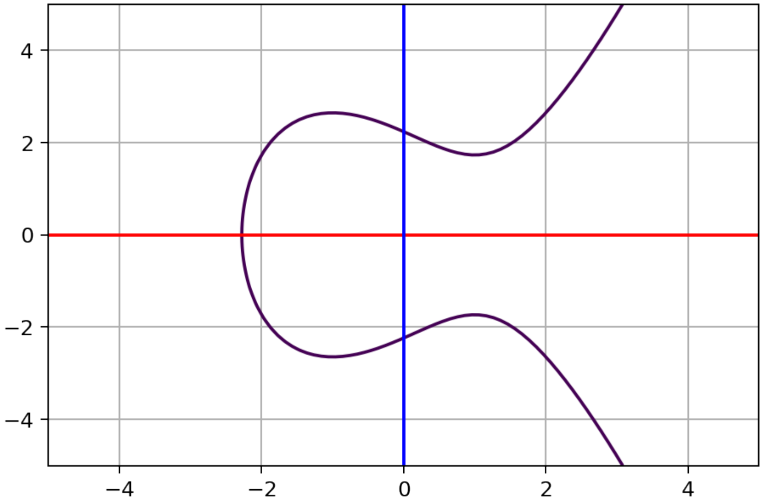
Elliptic Curve (definition)

An elliptic curve is defined by:

$$y^2 = x^3 + ax + b$$

(with some constraints of constants a and b)

Elliptic Curve for $y^2 = x^3 - 3x + 5$



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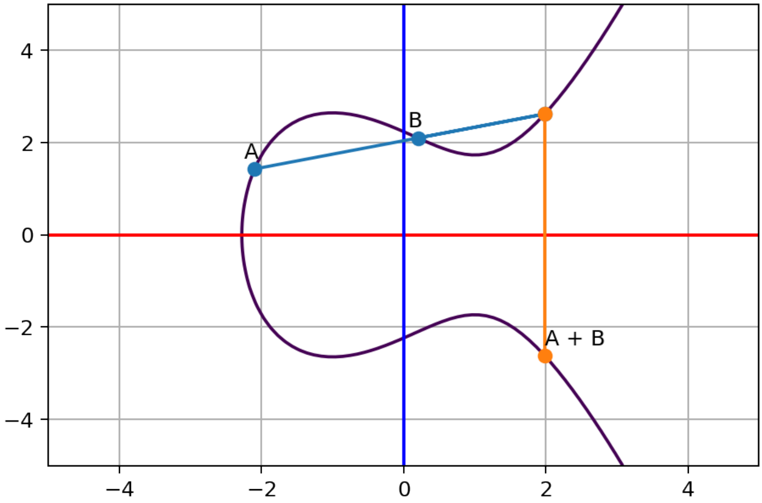
Addition Operation with an Elliptic Curve (definition)

Select two points on the curve, A and B, and draw a straight line through them. The line will intersect with the curve at a third point, R (and no other points). The horizontal inverse of point R, is defined as the addition of A and B.

$$A + B = -R$$

Addition Operation on Elliptic Curve



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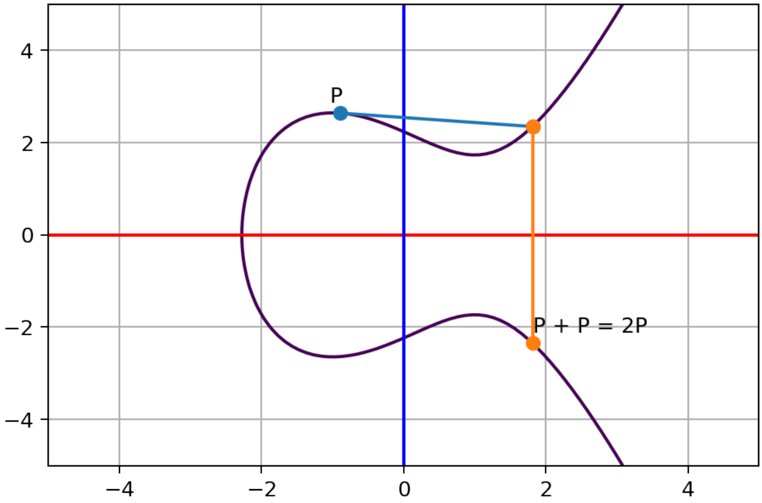
Self Addition on Elliptic Curve



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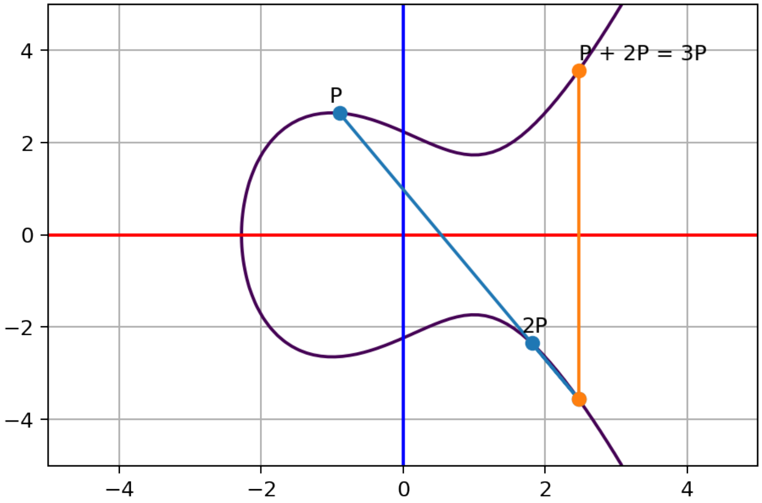
P + 2P on Elliptic Curve



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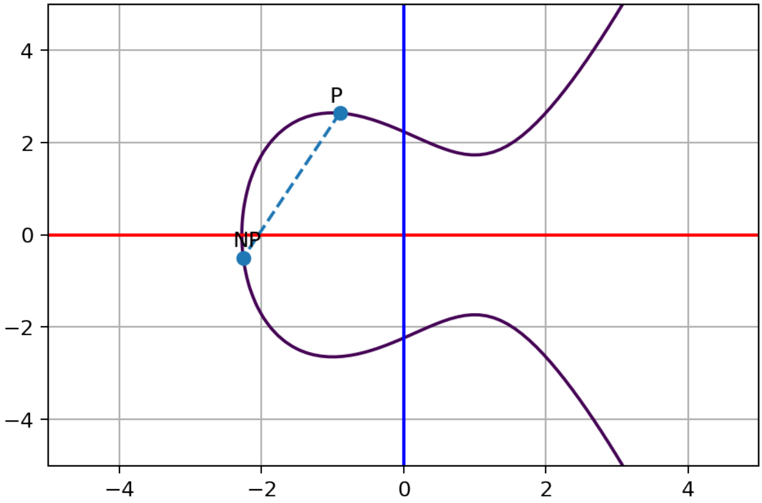
NP on Elliptic Curve



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How is Point Addition used in Elliptic Curve Cryptography?

- User chooses a point P (global public parameter)
- User chooses a large, random N (private key)
- User calculates NP (public key)
  - Easy, since there is a shortcut (described shortly)
- Challenge for attacker: given NP, find N
  - Computationally hard for large N

Shortcut for Calculating NP

- Assume N is large, e.g. 256-bit random number
- Naive point addition: $P + P + P + P + \ldots + P + P$ ($2^{256} - 1$ additions)
- Shortcut algorithm for point addition:
  - Calculate P, $P + P = 2P = 2^1 P$, $2P + 2P = 4P = 2^2 P$, $4P + 4P = 8P = 2^3 P$, $\ldots$, $2^{255} P$ (255 additions)
  - Write N as binary expansion, e.g.:
    - $N = 233 = 2^7 + 2^6 + 2^5 + 2^3 + 2^0$
    - $NP = 2^7 P + 2^6 P + 2^5 P + 2^3 P + 2^0 P$
    - In this example, there are 4 point additions
    - Maximum number of point additions for 256-bit N is 255
  - Calculate NP using the binary expansion
  - Maximum number of point additions for 256-bit N: $255 + 255 = 510$

Elliptic Curve with Modular Arithmetic

- The above discussed a normal elliptic curve
- But to ensure all values contained within finite coordinate space, modular arithmetic is used
- $y^2 \bmod p = (x^3 + ax + b) \bmod p$
- p is a prime number

Contents

Applications of ECC

- Secret key exchange, e.g. ECDH, ECMQV
- Digital signatures, e.g. ECDSA, EC-KCDSA
- Public key encryption, e.g. ECIES, PSEC

Elliptic Curve Diffie-Hellman Key Exchange (algorithm)

Assume users A and B have EC key pairs: $PU_A = NP$, $PR_A = N$, $PU_B = MP$, $PR_B = M$.

1. User A calculates secret $S_A = N \cdot PU_B = NMP$ using shortcut point addition.
2. User B calculates secret $S_B = M \cdot PU_A = MNP$ using shortcut point addition.

Choosing Parameters for ECC

- Parameters for ECC are usually standardised
  - Base point, P (also referred to as generator, G)
  - Curve parameters, a and b
  - Prime, p
  - Other parameters also included
- Common curves (see also `https://safecurves.cr.yp.to/`):
  - NIST FIPS 186: P-256, P-384 and 13 others
  - SECG: secp160k1, secp160r1, ... (NIST curves are a subset)
  - ANSI X9.62: prime192, prime256, ...
  - Other curves: Curve25519, Brainpool

Contents