Hidden vs. Exposed Terminal Problem in Ad hoc Networks

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Abstract

An ad hoc network is a collection of wireless mobile nodes that are capable of forming a co-operative agreement (network) between themselves without requiring any centralized control function. Due to their non-reliance on fixed infrastructure, ad hoc networks are gaining popularity in several networking applications including, military, rescue operations and meetings and conventions. Generally, protocols used at medium access and physical layers of ad hoc networks are similar to those used in infrastructure based wireless networks. It is a common practise in infrastructure based wireless networks to use pre-data exchange of control information to eliminate the "hidden terminal" problem. Although the use of pre-data exchange of control information introduces the "exposed terminals", it has been shown that throughput performance of infrastructure networks generally improves with use of such mechanisms. However, ad hoc networks topologies differ significantly from those of access point based (infrastructure) wireless networks, hence the effect of hidden ad exposed terminals in ad hoc networks is different than in infrastructure based networks. In this paper we show, through analytical studies and simulations, that exposed terminal problem dominates in ad hoc networks, and therefore, contrary to some previously published results, the use of RTS/CTS handshake may be detrimental to the network's throughput performance.

1 Introduction

Due to the lack of a centralised control entity in ad hoc networks, sharing of wireless bandwidth among ad hoc nodes (medium access control) must be organised in a decentralised manner. Therefore distributed Medium Access Control (MAC) mechanisms such as Carrier Sense Multiple Access with Collision Avoidance and its' variants such as MACA [1], MACA for Wireless (MACAW) [2] and 802.11 Distributed Coordination Function (DCF) [3] have gained widespread popularity in ad hoc networks. However, all these CSMA/CA based MAC protocols suffer from the well known "hidden terminal" problem.

In wireless networks, it is a commonly accepted practice to use a pre-data control information exchange (virtual medium sensing) to avoid the hidden terminal problem. One such virtual sensing mechanism is the 802.11 Request To Send/Clear To Send (RTS/CTS) exchange resulting in nodes getting exclusive access to the channel for a well-defined time period. However, the use of RTS/CTS-like schemes introduces the "exposed terminal problem"[1], where some nodes that heard the RTS/CTS exchange refrain from transmission even though they would not have interfered with any ongoing transmission.

The hidden terminal problem was well studied for access points (infrastructure) based networks [5] and it was shown that the introduction of virtual sensing (like RTS/CTS) improves their performance [1, 2, 4]. Ad hoc networks have gained popularity among researchers within the last decade, especially in military and emergency service provision contexts. Due to technical and commercial reasons, essentially the same distributed MAC layer protocols used in infrastructure based wireless networks have been considered for ad hoc networks.

However, ad hoc networks have significantly different topologies compared with the infrastructure based networks, leading to the question whether RTS/CTS like schemes have the same effect in both types of networks. In this paper, we discuss the hidden and exposed terminal problems in terms of the number of hidden and exposed terminals potentially affecting a communicating pair of nodes in a large ad hoc network with uniform node density.

The rest of this paper is organised as follows. Section 2 presents the definitions of hidden and exposed terminals in the context of wireless networks.

[1]The definition of "exposed terminals" in this paper may differ from other well-known definitions such as the one given in [4]. We define the exposed terminal problem in Section 2.2.

Furthermore, in this section we formulate the "hidden vs. exposed" terminal problem in terms of number of hidden and exposed terminals. Section 3 derives the number of hidden and exposed terminals potentially affecting a communicating pair of nodes in a large ad hoc network with uniform node density. This is followed by simulation studies, which show that the network throughput degrades with RTS/CTS mechanisms in large 802.11 based wireless ad hoc networks. In the last section we present our concluding remarks as well as suggestions for further studying the relative effect of hidden and exposed terminals in ad hoc networks.

2 Definitions and Problem Formulation

In this section, we define the hidden and exposed terminal problems. Furthermore, we present our framework for comparing the effects of hidden and exposed terminals in ad hoc network. We assume that all nodes have the same transmission and reception ranges.

2.1 The Hidden Terminal Problem

Figure 1 illustrates the hidden terminal problem. Suppose that node A wants to transmit to node B located at a distance x from A. By only sensing the medium, node A will not be able to hear transmissions by any node (C) in the dashed area denoted by $A(x)$, and will start transmitting, leading to collisions at node B. This is the well known hidden terminal problem, where the hidden nodes are located in the area $A(x)$.

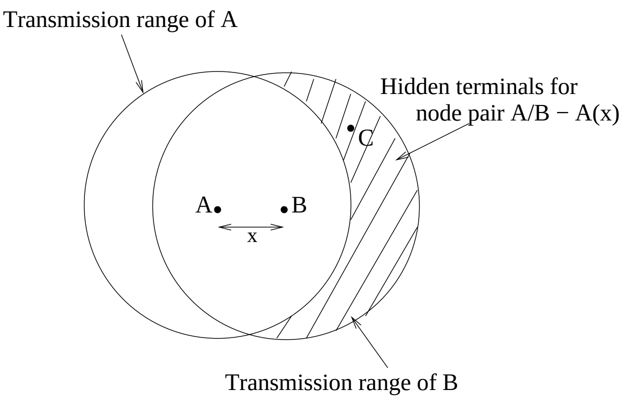

Figure 1. The hidden terminal problem

2.2 The Exposed Terminal Problem

RTS/CTS handshake mechanism was introduced to wireless MAC layers to eliminate the hidden terminal problem. However, this mechanism introduces a new problem termed the exposed terminal problem. We assume here an RTS/CTS exchange so that the issue of hidden terminal is addressed. Let us consider Figure 2 and assume that node A wants to transmit to node B.

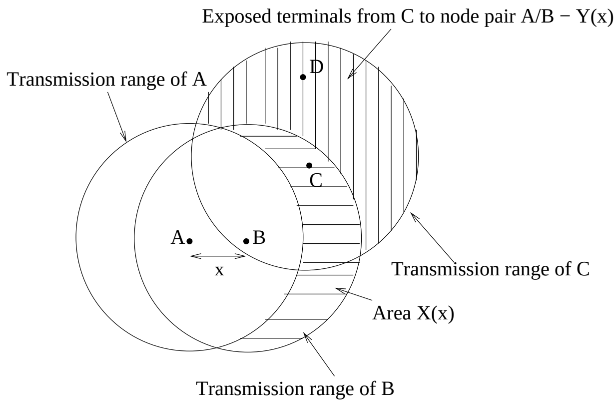

Figure 2. The exposed terminal problem

Node A sends an RTS and waits for B to send a CTS. Suppose a node D located in area $Y(x)$ wants to transmit data to node C located in area $X(x)$, and D transmits a RTS to C just before A sends the RTS to B. After receiving the RTS from D, C transmits a CTS. This CTS is heard by B upon which B will enter a backoff period preventing B from sending the CTS to A. Therefore, any transmission from a node within the area Y to a node within $X(x)$ will prevent A from transmitting data to B, although simultaneous transmissions from area $Y(x)$ to $X(x)$ would not have interfered with transmission from A to B. We define the terminals in the region $Y(x)$ as the exposed terminals for the node pair A/B. In this case, the number of transmissions that could occur between nodes from area $X(x)$ and nodes from area $Y(x)$ can be expressed as XY.

2.3 Hidden vs. Exposed Terminal Problem

In this paper, we evaluate the relative effects of hidden and exposed terminal problems through comparison of the number of such terminals affecting a given node in an ad hoc network. Let us consider the following arguments:

- In the case of hidden terminal problem, unsuccessful transmissions result from collisions between a transmission originated by a node such as A which cannot hear the on going transmissions to its corresponding node B. The probability of such a collision is proportional to the total number of terminals hidden from A.

- In the case of exposed terminal, unsuccessful transmissions result from nodes such as A being prevented from transmitting, because their corresponding node is unable to send a CTS. Again such unsuccessful transmissions are proportional to the number of exposed terminals. Both these events lead to degradation of a node's throughput.

In sections 2.1 and 2.2, we defined the number of hidden and exposed terminals for a given node pair A/B. Therefore, by appropriate integration we could calculate the average number of hidden and exposed nodes for a given node. Let us assume that during time period t, each node has the same probability, p, of sending a packet to a node in its vicinity. Then due to the above arguments, the ratio of hidden to exposed terminals is proportional to the relative degradation of a node's throughput due to hidden and exposed terminal effects.

For the hidden terminal case, it is enough for a nodes in region $A(x)$ (Figure 1) to transmit to any node in its vicinity. However, in the exposed terminal problem node D in region $Y(x)$ should send a RTS to C for the node B to hear the CTS send by C[2]. Therefore the effect of exposed terminals has to be further weighted by the average probability of a given node (D) in region $Y(x)$ communicating with a given node, (C) in region $X(x)$.

3 Effects of Hidden and Exposed Terminals

In this section, we derive the number of hidden and exposed terminals affecting a given node in an ad hoc network.

3.1 Hidden Terminals

The shaded region (with area $A(x)$ in Figure 1) shows the number of hidden terminals affecting node A. By using basic geometric arguments and assuming that the transmission range is R, it can be shown that:

$$A(x) = \pi R^2 - 2R^2 \arccos\left(\frac{x}{2R}\right) + \frac{x}{2}\sqrt{4R^2 - x^2} \tag{1}$$

Note that $A(x)$ only corresponds to the area containing nodes hidden from A when A wants to transmit to B. In order to find the total number of nodes potentially hidden from A, one needs to consider all possible nodes within the transmission range of node A.

[2]There are other nodes in the region $A(x)$ that would send CTS that could be heard by B. These nodes are taken into account during the integration process over the region $X(x)$.

Let us assume a uniform distribution of σ nodes per unit area in the network. Then $A(x)\sigma$ is the number of hidden terminals for a node pair A/B, when B is located within an annulus dx from distance x from A. Furthermore the probability of finding B in this annulus is:

$$\frac{2\pi x dx}{\pi R^2}$$

Therefore, the average number of terminals, H, that could be hidden from A, at any time A wants to transmit is:

$$\begin{aligned} H &= \frac{2}{R^2}\int_0^R xA(x)dx \\ &= 1.3\sigma R^2 \end{aligned} \tag{2}$$

3.2 Exposed Terminals

In this section, we derive an expression for the number of exposed terminals for a given node in an ad hoc network. Let us consider Figure 3:

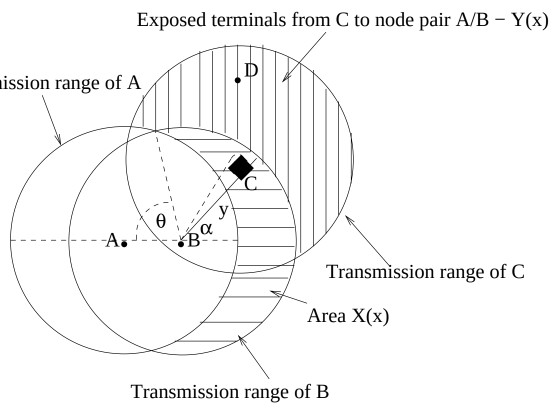

Figure 3. Exposed terminal calculation

Using basic geometry, the notations presented in Figure 3, symmetry of the system and the arguments presented in Section 3.1 it can be shown that:

$$Y(x, y, \alpha) = \pi R^2 - 2R^2 \arccos\left(\frac{y}{2R}\right) + \frac{y}{2}\sqrt{4R^2 - y^2} \tag{3}$$

In Figure 4 it can be observed that the expression for area Y could take different forms depending on the position of node C within the area $X(x)$.

If node C is located outside the circle centred around point T, (intersection of transmission ranges of A and B), the area Y can be expressed by the Equation (3). However if the node C is located out side this circle then the area for exposed terminal is given by the expression:

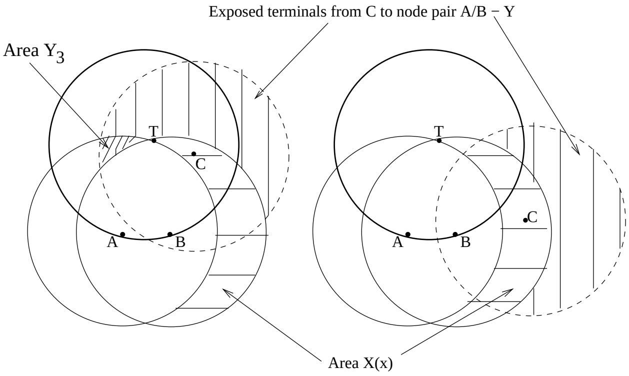

Figure 4. Various scenarios for exposed terminal calculations

$$\pi R^2 - 2R^2 \arccos\left(\frac{y}{2R}\right) + \frac{y}{2}\sqrt{4R^2 - y^2} - Y_3(x, y, \alpha)$$

Then from Figure 3 the number of exposed terminals can be found by integrating over the region XY as follows:

$$\frac{2}{R^2}\int_0^R x \int_0^{\pi-\theta} \int_{r_1}^R 2Y(x, y, \alpha)\, \sigma^2 y \, dy \, d\alpha \, dx \quad (4)$$

Where

R - Transmission/reception range of the nodes

σ - Node density of the network

$\theta = \arccos\left(\frac{x}{2R}\right)$

$r_1 = \sqrt{R^2 - x^2 \sin^2 \alpha} - x \cos \alpha$

$$\begin{aligned} Y(x, y, \alpha) &= \pi R^2 - 2R^2 \arccos\left(\tfrac{y}{2R}\right) \\ &\quad + \tfrac{y}{2}\sqrt{4R^2 - y^2} \\ &= Y_1(x, y, \alpha) \end{aligned}$$

or

$$\begin{aligned} Y(x, y, \alpha) &= \pi R^2 - 2R^2 \arccos\left(\tfrac{y}{2R}\right) \\ &\quad + \tfrac{y}{2}\sqrt{4R^2 - y^2} - Y_3(x, y, \alpha) \\ &= Y_2(x, y, \alpha) \end{aligned}$$

depending on the position of C.

Although the integration presented in equation (4) is not tractable, a lower bound for this can be derived. From this derivation, presented in Appendix A, it can be shown that the number of exposed terminals for node A, E, is:

$$E > 1.03\sigma^2 R^4 \quad (5)$$

In Section 2.3 we explained that for the exposed terminal case it is necessary for node D in region Y to communicate with node C in region X (refer to Figure 2) to experience the effect of exposed terminals. Therefore we should weight the number of exposed terminals given by (18) by the average probability of the event that a given node in the region Y will be communicating with a given node in region X.

Assume each node communicates with all other nodes in its' vicinity with the same probability. Then the probability of Node D communicating with C (refer to Figure 2) is:

$$\frac{\sigma y d\alpha dy}{\sigma \pi R^2}$$

Then the average probability of this event is:

$$\begin{aligned} P_{\mathrm{av}} &= \frac{2}{R^2}\int_0^R 2x \int_0^{\pi-\theta} \int_{r_1}^R \frac{y}{\pi R^2}\, dy \, d\alpha \, dx \\ &= \frac{2}{\pi R^6}\int_0^R x \left(S(x^3 - xR^2) + 2R^4 \arctan\left(\frac{xS}{2R^2 - x^2}\right)\right) dx \\ &= 0.28 \end{aligned}$$

Where r_1, θ and S are as defined before.

On the other hand if we assume that a given node D has a probability 1 of communicating with given node C, then we can estimate that the number of average exposed terminals for a given node is between $0.29\sigma^2 R^4$ and $1.03\sigma^2 R^4$.

As we are interested in the lower bound on the number of exposed terminals we can conclude that the number of exposed terminals affecting a node in an ad hoc network is greater than $0.29\sigma^2 R^4$. Figure 5 shows the number of hidden terminals and the lower bounds on the number of exposed terminals for various node densities.

It can be observed in Figure 5 that for densities greater than 4 nodes per transmission neighbourhood, the number of exposed terminals dominate the number of hidden terminals. Assuming that the effect (degradation of performance) of hidden and exposed terminals on the network performance is proportional to the number of hidden and exposed terminals perceived by a pair of nodes (see Section 2.3), we can conclude that for most topologies of large ad hoc networks the exposed terminal effect dominates the hidden terminal effect. It has to be emphasise here that this result is only applicable to networks large enough, such that the majority of the nodes have the full influence of exposed terminals. In other words, the majority of the nodes should be at least 3 times the transmission range away from all the boundaries of the network. In applications of ad hoc networks, such as military networks or conventions generally contain a large number of nodes. Therefore it beneficial to operate these ad hoc network without the RTS/CTS

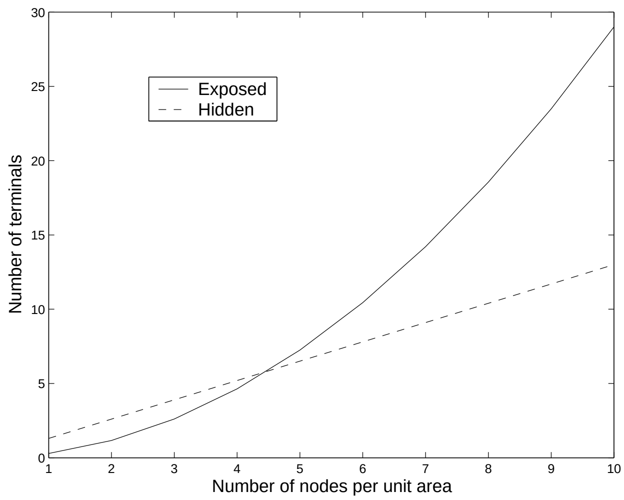

Figure 5. Variation of hidden and exposed terminals with node density

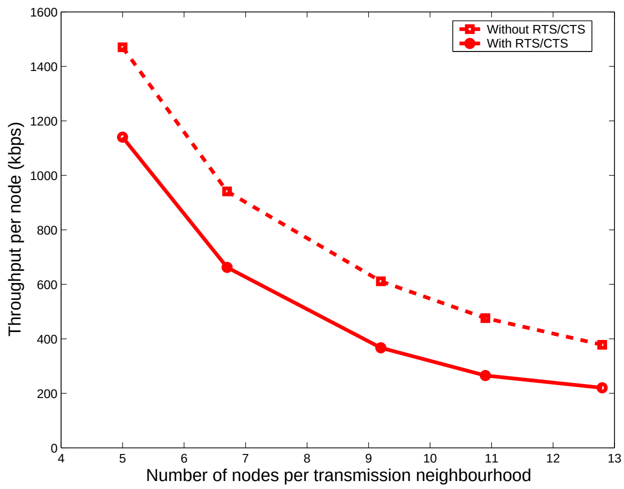

Figure 6. Throughput performance with and without RTS/CTS

mechanism to eliminate the performance degradation caused by the exposed terminals.

4 Simulations

Network simulations using OPNET network simulator were conducted to further establish the finding presented in section 3. The nodes were distributed in a grid pattern evenly throughout the network. In these simulations the average throughput of nodes in the centre of the network was calculated for different node densities with and without the RTS/CTS mechanism. A network with stationary nodes, each communicating only with the neighbouring nodes using 802.11 medium access control layer and Dynamic Source Routing (DSR) was used in the simulation. Figure 6 shows the node throughput for different scenarios.

It can be seen in Figure 6 that the throughput at the centre of the network can be improved on average by 50% without RTS/CTS. This agrees with the analytical findings that the performance is degraded by the exposed terminals compared to the effect of hidden terminals. Therefore, by combing the results of analytical and simulation studies it can be concluded that the throughput of wireless ad hoc networks can be improved by not using handshake mechanisms such as RTS/CTS.

5 Conclusions

In this paper, we evaluated the relative throughput performance of ad hoc networks with and without pre-data control schemes, such as RTS/CTS mechanism in 802.11 system. The analytical and simulation results suggest that throughput performance degrades due to use of RTS/CTS like predata handshake mechanisms. This argues strongly against the use of RTS/CTS in ad hoc networks, contrary to the widely published results [6, 7, 8].

It is emphasised here that Figure 5 shows the variation of number of hidden and exposed terminals, not the effect of hidden and exposed terminals on the performance of ad hoc networks. Therefore this study does not take into account all the effects of hidden and exposed terminals on the transmission patters of ad hoc networks. Furthermore we do not take into account the effect of data rate or the condition of the physical channel in to consideration. A better method would be to evaluate the relative transmission opportunities and the relative success of these transmissions in ad hoc networks with and without the RTS/CTS scheme, which can take into account parameters such as the data rate and conditions of the physical channel. The transmission opportunities in the network in conjunction with the probability of packet loss due to collisions can then be directly related to the network throughput.

Acknowledgments

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Appendix A

In this appendix we provide the solution for the integration presented in equation (4). This derives the number of exposed terminals affecting a pair of communicating nodes in an ad hoc network.

In figure 7, assume that (α_1, r_1) and (α_2, R) are the intersection points for a circle centred around point T and circles centred around points A and B respectively (refer to Figure 4). Also let us assume that in the range $\alpha_1 < \alpha < \alpha_2$, $y = r_2$ is the distance between nodes B and C when C is located distance R away from point T, as shown in Figure 7. Therefore for $\alpha_1 < \alpha < \alpha_2$:

$$\begin{array}{lll} \text{If} \quad y > r_2 & Y(x,y,\alpha) & = Y_1(x,y,\alpha) \\ \text{otherwise} & Y(x,y,\alpha) & = Y_2(x,y,\alpha) \end{array}$$

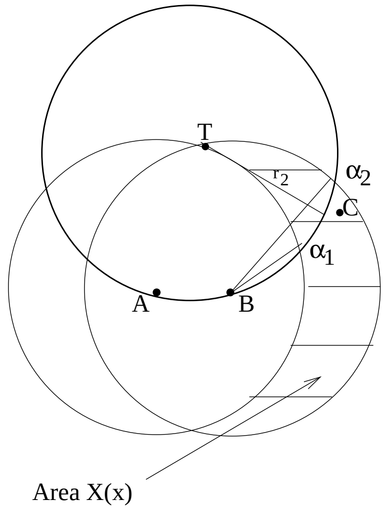

Figure 7. Parameters for exposed terminal calculations

Now let us consider the following integral $E(x)$:

$$\begin{aligned} E(x) &= \int_0^{\pi-\theta} \int_{r_1}^{R} Y(x,y,\alpha)\, d\alpha\, dy \\ &= \int_0^{\alpha_1} \int_{r_1}^{R} Y_1(y,\alpha)\, \sigma^2 y\, dy\, d\alpha + \int_{\alpha_1}^{\alpha_2} \int_{r_1}^{r_2} Y_2(y,\alpha)\, \sigma^2 y\, dy\, d\alpha \\ &\quad + \int_{\alpha_1}^{\alpha_2} \int_{r_2}^{R} Y_1(y,\alpha)\, \sigma^2 y\, dy\, d\alpha + \int_{\alpha_2}^{\pi-\theta} \int_{r_1}^{R} Y_2(y,\alpha)\, \sigma^2 y\, dy\, d\alpha \\ &= \int_0^{\pi-\theta} \int_{r_1}^{R} Y_1(y,\alpha)\, \sigma^2 y\, dy\, d\alpha - \int_{\alpha_1}^{\alpha_2} \int_{r_1}^{r_2} Y_3(y,\alpha)\, \sigma^2 y\, dy\, d\alpha \\ &\quad - \int_{\alpha_2}^{\pi-\theta} \int_{r_1}^{R} Y_3(y,\alpha)\, \sigma^2 y\, dy\, d\alpha \\ &= B(x) - C(x) - D(x) \end{aligned} \tag{6}$$

Where

$$\begin{aligned} \alpha_1 &= \tfrac{\pi}{6} \\ \alpha_2 &= \tfrac{2\pi}{3} - \arccos\left(\tfrac{x}{2R}\right) \\ r_1 &= \sqrt{R^2 - x^2 \sin^2\alpha} - x\cos\alpha \\ r_2 &= \sqrt{4R^2 - x^2}\sin\alpha - x\cos\alpha \end{aligned}$$

Let us consider $A(y)$, as defined in Equation 7

$$A(y) = \pi R^2 - 2R^2 \arccos\left(\frac{y}{2R}\right) + \frac{y}{2}\sqrt{4R^2 - y^2} - \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right) Ry \tag{7}$$

The first and second derivatives of $A(y)$ are given by:

$$A'(y) = \sqrt{4R^2 - y^2} - \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right) R \tag{8}$$

and

$$A''(y) = \frac{-y}{\sqrt{4R^2 - y^2}}$$

It can be shown that in the range $y = 0 \ldots R$, $y = \sqrt{4 - \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)^2} R = y_{\max}$ is the only solution for $A'(y) = 0$ and $A''(y_{\max}) < 0$. Therefore in the range $y = 0 \ldots R$, $A(y)$ has a local maxima at $y_{max} = \sqrt{4 - \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)^2} R$ and $A(y_{\max}) = 0.034$. Furthermore $A(0) = 0$ and $A(R) = 0$, implying that $A(y)$ is strictly non-negative in the range $0 \ldots R$.

Therefore,

$$Y_1(y, \alpha) > \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right) Ry \tag{9}$$

In this work, our objective is to show that the number of exposed terminals for a node is greater than the number of hidden terminals for the same node. As the integral (4) is not trivial to solve, we have decided to estimate a lower bound for (4) and show that this lower bound is greater than the number of hidden terminals given by Equation (2).

Using (9) and (6) it can be shown that:

$$\begin{aligned} B(x) &> 2KR\sigma^2 \int_0^{\pi-\theta} \int_{r_1}^{R} y^2 \, dy \, d\alpha \\ &= \frac{2KR\sigma^2}{3} \int_0^{\pi-\theta} R^3 - (\sqrt{R^2 - x^2 \sin^2 \alpha} - x \cos \alpha)^3 d\alpha \end{aligned} \tag{10}$$

Where $K = \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)$.

Using a argument similar to the one used the in previous section, it can be shown that for all $y = y_1 \ldots R$:

$$R^3 - (\sqrt{R^2 - x^2 \sin^2 \alpha} - x \cos \alpha)^3 > \frac{(\alpha - (\pi - \theta))(R^3 - (R - x)^3)}{\pi - \theta} \tag{11}$$

Therefore, by using Equations (10) and (11);

$$B(x) > \frac{KR\sigma^2}{3} (\pi - \theta)(R^3 - (R - x)^3) = B_{\text{lower}}(x) \tag{12}$$

From Figure 4 it can be observed that it is not trivial to calculate the area Y_3 for a given x, y and α. As we argued earlier, the objective of this study is to find the lower bound of the integral 4), which allows the substitution of the maximum of Y_3, $Y_{3,max}$, for Y_3 in the integral 6). Furthermore, it can be shown that:

$$Y_{3,\max}(x) = R^2 \sin\left(\frac{x}{2R}\right) \tag{13}$$

As now we have found a suitable expression for Y_3, we can now evaluate the terms $C(x)$ and $D(x)$ in Equation (6). Using Equations (6) and (13) and the arguments outlined in the previous paragraph, we can deduce that:

$$\begin{aligned} C(x) &< 2R^2 \sin\left(\frac{x}{2R}\right) \int_{\alpha_1}^{\alpha_2} \int_{r_1}^{r_2} \sigma^2 y \, dy d\alpha \\ &= R^2 \sigma^2 \sin\left(\frac{x}{2R}\right) \int_{\alpha_1}^{\alpha_2} r_2^2 - r_1^2 \, d\alpha \end{aligned} \tag{14}$$

Where r_1 and r_2 are as defined earlier. Using methods used in earlier in this section[3], it can be shown that:

$$\begin{aligned} C(x) &< \frac{R^2\sigma^2 \sin\left(\frac{x}{2R}\right)}{2} \left(\alpha_2 - \frac{\pi}{6}\right) [\Gamma - \Theta] \\ &= C_{\text{upper}}(x) \end{aligned} \tag{15}$$

Where $\alpha_2 = \frac{2\pi}{3} - \arccos\left(\frac{x}{2R}\right)$
$\Gamma = \left(\sqrt{4R^2 - x^2} \sin \alpha_2 - x \cos \alpha_2\right)^2$
$\Theta = \left(\sqrt{R^2 - x^2 \sin^2 \alpha_2} - x \cos \alpha_2\right)^2$

We can now find a lower limit for $D(x)$ by using (13):

$$\begin{aligned} D(x) &< 2R^2 \sin\left(\frac{x}{2R}\right) \int_{\alpha_2}^{\pi-\theta} \int_{r_1}^{R} \sigma^2 y \, dy d\alpha \\ &= \sigma^2 R^2 \sin\left(\frac{x}{2R}\right) \int_{\alpha_2}^{\pi-\theta} R^2 - \Lambda \, d\alpha \\ &= \frac{\sigma^2}{4} \sin\left(\frac{x}{2R}\right) \left[S\left(2x^3 - 3R^2 x\right) + \Phi + \sqrt{3} R^2 x^2\right] \\ &= D_{\text{upper}}(x) \end{aligned} \tag{16}$$

Where α_2, θ and r_1 are as defined before.

$S = \sqrt{4R^2 - x^2}$
$\Lambda = \left(\sqrt{R^2 - x^2 \sin^2 \alpha} - x \cos \alpha\right)^2$
$\Phi = 4R^4 \left(\arctan\left(\frac{xS}{2R^2 - x^2}\right) - \arctan\left(\frac{x}{S}\right)\right)$

Having found suitable expression for $B(x)$, $C(x)$ and $D(x)$, we can now evaluate the integral (6). Substituting from inequalities (12), (15) and (16);

$$E(x) < B_{\text{lower}}(x) - C_{\text{upper}}(x) - D_{\text{upper}}(x) \tag{17}$$

The number of potential exposed terminals for the node pair A/B is calculated by weighted integration of $E(x)$ from $x = 0 \ldots R$. Therefore, a lower

[3]By calculating the first and second derivatives of the appropriate function it can be shown that the corresponding function is strictly negative in the region $\alpha_1 < \alpha < \alpha_2$.

limit for the number of exposed terminals could be estimated by integrating both sides of the Equation (17):

$$\begin{aligned} E_{\text{lower}} &= \frac{2}{R^2}\int_0^R B_{\text{lower}}(x) - C_{\text{upper}}(x) - D_{\text{lower}}(x)\ dx \\ &= 1.03\sigma^2 R^4 \end{aligned} \tag{18}$$